Technical Appendix to
"Market Power and Efficiency in a Search Model"

Manolis Galenianos*, Philipp Kircher† and Gábor Virág‡

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This note provides the technical details for the proof that minimum wages are detrimental to efficiency (Proposition 4.2) in "Market Power and Efficiency in a Search Model" by the authors. For notation, please see that paper. We still have to show

Lemma 0.1 *Suppose that all low productivity firms offer a wage $w_2 < x_1$. Then in the game between the high productivity firms only, there is a unique symmetric equilibrium wage offer w_1.*

Proof. First, note that the result holds for the case of $m_1 = 1$, because in this case the game between the high productivity firms becomes a decision problem for the single high productivity firm and then the strict concavity of the profit function implies that there is a unique best reply for the high productivity firm.

Now, we turn to the case where $m_1 \geq 2$. A high productivity firm, firm i maximizes his profit with respect to the wage he offers. Since the profit function π_i is concave in w_i the necessary first order condition is sufficient for the optimum as well. Since w_i is one-to-one with the application probability achieved p_i, therefore one can take p_i as the choice variable of firm i when taking the wages posted by the other firms (low or high productivity) as given. Let $\widetilde{U}(p_i, w_1, w_2)$ denote the equilibrium utility of the workers if firm i posts a wage that yields an application probability of p_i for firm i if the other high productivity firms posted wage w_1 and the low productivity firms posted wage w_2. The profit of firm i can be written as

$$\pi_i = x_1 H(p_i) - n p_i \widetilde{U}(p_i, w_1, w_2).$$

*Corresponding author. Pennsylvania State University. E-mail: manolis@psu.edu.
†University of Pennsylvania and IZA. E-mail: kircher@econ.upenn.edu.
‡University of Rochester. E-mail: gvirag@mail.rochester.edu.

Let w_1 denote the wage posted by the other high productivity firms and w_2 the wage posted by the low productivity firms and let $p_1(p_i)$ and $p_2(p_i)$ denote the application probabilities that those other firms obtain if firm i offers a wage that secures him an application probability of p_i. Following the proof (and the notation) of Proposition 4.1, it follows that

$$\frac{\partial p_2}{\partial p_i} = \frac{-1/\rho_2}{\frac{m_1-1}{\rho_1} + \frac{m_2}{\rho_2}}.$$

Let us now study the behavior of function $\widetilde{U}(p_i, w_1, w_2)$. By the indifference condition of the workers it holds that

$$\widetilde{U}(p_i, w_1, w_2) = w_2 G(p_2(p_i)) = w_1 G(p_1(p_i)).$$

Therefore,

$$\frac{\partial \widetilde{U}}{\partial p_i} = w_2 g(p_2) \frac{-1/\rho_2}{\frac{m_1-1}{\rho_1} + \frac{m_2}{\rho_2}}.$$

Take an equilibrium and denote the application probability received by the high productivity firms as p_1. In the notation of Proposition 4.1 it means that $w_1 = w_1^*(p_1, w_2)$. We are ready to state the first order necessary (and, by concavity, sufficient) condition of optimum for firm i's problem for a symmetric equilibrium where $p_i = p_1$:

$$x_1 h(p_1) = n\widetilde{U}(p_1, w_1^*(p_1, w_2), w_2) - np_1 w_2 g(p_2) \frac{1/\rho_2}{\frac{m_1-1}{\rho_1} + \frac{m_2}{\rho_2}}.$$

Note, that in a symmetric equilibrium $p_2 = \frac{1-m_1p_1}{m_2}$ and thus (fixing w_2)

$$\nu = n\widetilde{U}(p_1, w_1^*(p_1, w_2), w_2) - np_1 w_2 g(\frac{1-m_1p_1}{m_2}) \frac{1/\rho_2}{\frac{m_1-1}{\rho_1} + \frac{m_2}{\rho_2}}$$

is a function of p_1 only. Since h is decreasing in p_1 it is sufficient to show that α is increasing in p_1 to have that the necessary first order condition has a unique solution, which would conclude the proof.

The rest of the proof establishes that ν is an increasing function of p_1. First, note that

$$\widetilde{U}(p_1, w_1^*(p_1, w_2), w_2) = w_2 G(\frac{1-m_1p_1}{m_2}).$$

Therefore, we need to prove that

$$\alpha(p_1) = G(\frac{1-m_1p_1}{m_2}) - p_1 g(\frac{1-m_1p_1}{m_2})\frac{1}{(m_1-1)\frac{\rho_2}{\rho_1}+m_2}$$

is increasing in p_1. Let us introduce the shorthand notations $G_2 = G(\frac{1-m_1p_1}{m_2})$ and $g_2 = g(\frac{1-m_1p_1}{m_2})$ and

$$T = \frac{1}{(m_1-1)\frac{\rho_2}{\rho_1}+m_2}$$

and thus

$$\alpha = G_2 - p_1 g_2 T.$$

Recalling that $\rho_2 = \frac{g_2}{G_2}$ yields

$$\alpha' = g_2(\frac{-m_1}{m_2} - T + p_1T^2(m_1-1)(\frac{\rho_2}{\rho_1})' + \rho_2 p_1 T\frac{m_1}{m_2}) - p_1 T G_2 \rho_2'.$$

Now note that $g_2 < 0$. It holds that

$$\rho_2' = \rho'(p_2)(\frac{-m_1}{m_2}),$$

where $\rho(x) = \frac{g(x)}{G(x)}$. At the end of the proof we show that ρ is an increasing function and thus $\rho_2' < 0$ holds when $n \geq 5$. Therefore, it is sufficient for $\alpha' > 0$ if

$$\frac{-m_1}{m_2} - T + p_1T^2(m_1-1)(\frac{\rho_2}{\rho_1})' + \rho_2 p_1 T\frac{m_1}{m_2} < 0$$

holds for all $p_1 \geq \frac{1}{m_1+m_2}$. This last condition can be rewritten as

$$(\frac{\rho_2}{\rho_1})' < \frac{\frac{m_1}{m_2} + T - \rho_2 p_1 T\frac{m_1}{m_2}}{p_1T^2(m_1-1)}.$$

After substituting and using that $p_1 \leq \frac{1}{m_1}$ it follows that

$$\frac{\frac{m_1}{m_2} + T - \rho_2 p_1 T\frac{m_1}{m_2}}{p_1T^2(m_1-1)} > \frac{\rho_2}{\rho_1}[\frac{m_1}{p_1m_2}(m_1-1+m_2) + m_1 - \rho_2\frac{m_1}{m_2}].$$

On the other hand it holds that

$$(\frac{\rho_2}{\rho_1})' = \frac{\rho_2}{\rho_1}(\frac{\rho_2'}{\rho_2} - \frac{\rho_1'}{\rho_1})$$

and thus it is sufficient to prove that for all $p_1 \in [\frac{1}{m_1+m_2}, \frac{1}{m_1}]$ it holds that

$$\frac{m_1}{p_1 m_2}(m_1 - 1 + m_2) + m_1 - \rho_2 \frac{m_1}{m_2} \geq \frac{\rho_2'}{\rho_2} - \frac{\rho_1'}{\rho_1}.$$

The rest of the proof establishes that

$$-\frac{\rho_1'}{\rho_1} < \frac{m_1}{p_1 m_2}(m_1 - 1 + m_2) + m_1 \tag{1}$$

and that

$$\frac{\rho_2'}{\rho_2} < -\rho_2 \frac{m_1}{m_2}, \tag{2}$$

which would yield the result. First, we establish (2). More precisely, we need to show that for all $p_2 \leq \frac{1}{m_1+m_2}$

$$\frac{\rho_2'}{\rho_2} = \frac{\rho'(p_2)}{\rho(p_2)}(\frac{-m_1}{m_2}) < -\rho(p_2)\frac{m_1}{m_2}$$

or that

$$\rho'(p_2) < \rho^2(p_2). \tag{3}$$

Noting that

$$\rho(x) = \frac{n(1-x)^{n-1}}{1-(1-x)^n} - \frac{1}{x}$$

yields that formula (3) is equivalent to

$$2 \leq \tau(x) = (n+1)x + ((n-1)x + 2)(1-x)^n.$$

It is immediate that $\tau(0) = 2$ and $\tau'(0) = 0$. Moreover, for all $x > 0$ it holds that

$$\tau''(x) = (n+1)(n-1)(1-x)^{n-2} nx > 0$$

and thus for $x \in (0, 1]$ it holds that $\tau(x) > 2$, which establishes formula (3).

To establish (1), note that $\frac{\rho_1'}{\rho_1} \leq 0$, because $\rho_1 < 0$ and $\rho_1' > 0$. First, we show that for all x

$$-\frac{\rho'(x)}{\rho(x)} \leq \frac{1}{x}. \tag{4}$$

This can be rewritten as

$$(\rho(x)x)' \leq 0. \tag{5}$$

But it holds that

$$\rho(x)x = \frac{nx(1-x)^{n-1}}{1-(1-x)^n} - 1$$

and thus (5) indeed holds after taking derivatives of this expression. Note, that it must hold by construction that $p_1 \leq \frac{1}{m_1}$ and thus

$$\frac{m_1}{p_1 m_2}(m_1 - 1 + m_2) + m_1 > (m_1)^2 + m_1.$$

Therefore, (4) implies that

$$p_1 \geq \frac{1}{(m_1)^2 + m_1} \Rightarrow -\frac{\rho_1'}{\rho_1} \leq (m_1)^2 + m_1 < \frac{m_1}{p_1 m_2}(m_1 - 1 + m_2) + m_1,$$

as needed. Therefore, suppose that $p_1 \leq \frac{1}{(m_1)^2 + m_1}$ and thus

$$\frac{m_1}{p_1 m_2}(m_1 - 1 + m_2) + m_1 > (m_1)^3 + m_1.$$

Then (4) implies that

$$p_1 \geq \frac{1}{(m_1)^3 + m_1} \Rightarrow -\frac{\rho_1'}{\rho_1} \leq (m_1)^3 + m_1 < \frac{m_1}{p_1 m_2}(m_1 - 1 + m_2) + m_1,$$

as needed. Therefore, if $p_1 \geq \frac{1}{(m_1)^3 + m_1}$, then formula (1) is established. By using this argument iteratively, one can handle all cases where $p_1 \geq \frac{1}{(m_1)^n + m_1}$ for all positive integer n. However, in any equilibrium, it must hold that $p_1 \geq \frac{1}{m_2 + m_1}$, because more productive firms obtain a higher probability of visit, than less productive firms under our assumptions. Note, that if $(m_1)^k \geq m_2$, then

$$p_1 \geq \frac{1}{m_2 + m_1} \Rightarrow p_1 \geq \frac{1}{(m_1)^k + m_1}$$

and thus the proof is completed for that case. However, since $m_1 \geq 2$ there exists such a k that $(m_1)^k \geq m_2$ and thus formula (1) is indeed established.

Now, we prove that for all $n \geq 5$ it holds that

$$\rho_n(x) = \frac{g_n(x)}{G_n(x)} = \frac{n(1-x)^{n-1}}{1-(1-x)^n} - \frac{1}{x}$$

is an increasing functions in x. We adopt an inductive argument for all values of $n > 5$, using $n = 5$ as the induction anchor that can be verified in a straightforward manner. Let $x > y$ and we want to establish that if $\rho_n(x) > \rho_n(y)$ then $\rho_{n+1}(x) > \rho_{n+1}(y)$ as well. Since all of these values are negative it is equivalent to establish that if $0 < \frac{\rho_n(x)}{\rho_n(y)} < 1$ then $0 < \frac{\rho_{n+1}(x)}{\rho_{n+1}(y)} < 1$

as well. In turn, for this it is sufficient to show that

$$\frac{(n+1)(1-x)^n}{1-(1-x)^n} - \frac{n(1-x)^{n-1}}{1-(1-x)^n} < \frac{(n+1)(1-y)^n}{1-(1-y)^n} - \frac{n(1-y)^{n-1}}{1-(1-y)^n}.$$

If one considers n as a continuous variable for the purpose of the proof, then it is sufficient to establish that

$$\frac{\partial^2}{\partial n \partial x}\frac{n(1-x)^{n-1}}{1-(1-x)^n} > 0.$$

This last condition is equivalent to

$$\gamma(x) = (1-(1-x)^n)^2 - 2n(1-(1-x)^n) - (n+n^2)\ln(1-x)(1-x)^n - (n^2-n)\ln(1-x) > 0.$$

For all n it holds that $\gamma(0) = 0$ and thus it is sufficient to have $\gamma'(x) > 0$ for all $x \in (0,1]$, which is equivalent to

$$\delta(x) = (1 - \frac{1}{(1-x)^n})\,(1-n) + (2 - 2\,(1-x)^n) + (\ln(1-x))\left(n+n^2\right) > 0$$

for all $x > 0$. Again, $\delta(0) = 0$ and thus it is sufficient to have $\delta'(x) > 0$ for all $x \in [0,1]$, which is equivalent to

$$\varepsilon(x) = 2\,(1-x)^{2n} - (1-x)^n\,(1+n) + n - 1 > 0.$$

Again, $\varepsilon(0) = 0$ and thus it is sufficient to have $\varepsilon'(x) > 0$ for all $x \in (0,1]$, which is equivalent to

$$(1+n) - 4\,(1-x)^n > 0,$$

which holds, because $1+n > 4$ and $1 \geq (1-x)^n$ for all $x \in (0,1]$ and $n \geq 5$.

Finally, we need to cover the case when $n < 5$, in which case function ρ is not increasing in x. In this case the relevant first order condition is a polynomial in degree n and thus one can use standard techniques to prove that there is a unique solution for p_1 in the relevant range. ■